ON QUANTUM QUADRATIC OPERATORS OF $\mathbb{M}_2(\mathbb{C})$ AND THEIR DYNAMICS

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Abstract. In the present paper we study nonlinear dynamics of quantum quadratic operators (q.q.o) acting on the algebra of 2×2 matrices $\mathbb{M}_2(\mathbb{C})$. First, we describe q.q.o. with Haar state as well as quadratic operators with the Kadison-Schwartz property. By means of such a description we provide an example of positive q.q.o. which does not the Kadision-Schwartz operator. Then we study stability of dynamics of q.q.o.

Mathematics Subject Classification: 46L35, 46L55, 46A37. 60J99.
Key words: quantum quadratic operators; quadratic operator; Kadison-Schwartz operator; nonlinear dynamics; stability.

1. Introduction

It is known that the theory of Markov processes is a rapidly developing field with numerous applications to many branches of mathematics and physics. However, there are physical systems that can not be described by Markov processes. One of such systems is given by quadratic stochastic processes (see [2]). Quadratic stochastic processes are related to the notion of a quadratic stochastic operator, which was introduced in [1]. The problem of studying the behavior of trajectories of quadratic stochastic operators was stated in [21]. The limit behavior and ergodic properties of trajectories of quadratic stochastic operators were studied in [7, 9, 10, 14, 22]. However, such kind of operators do not cover the case of quantum systems. Therefore, in [4, 5] quantum quadratic operators acting on a von Neumann algebra were defined and studied. Certain ergodic properties of such operators were studied in [15, 16]. In those papers basically dynamics of quadratic operators were defined due to some recurrent rule which makes a possibility to study asymptotic behaviors of such operators. However, with a given quadratic operator one can define also a non-linear operator whose dynamics (in non-commutative setting) are not studied yet. Very recently, in [6] convergence of ergodic averages associated with mentioned non-linear operator are studied by means of absolute contractions of von Neumann algebras. Actually, it is not investigated nonlinear dynamics of convolution operators. Therefore, a complete analysis of dynamics of quantum quadratic operator is not well studied.

In the present paper we are going to study nonlinear dynamics of quantum quadratic operators acting on the algebra of 2×2 matrices $\mathbb{M}_2(\mathbb{C})$. Since positive, trace-preserving maps arise naturally in quantum information theory (see e.g. [18]) and other situations in which one wishes to restrict attention to a quantum system that should properly be considered a subsystem of a larger system with which it interacts. Therefore, after preliminaries (Sec. 2) in section 3, we describe quadratic operators with Haar state (invariant with respect to trace), namely certain characterizations of q.q.o, Kadison-Schwartz operators, which are invariant w.r.t. trace, are given. By means of such a description in Section 4, we shall provide an example of positive q.q.o. which is not a Kadision-Schwartz operator. On the other hand, the such a characterization is related to the separability condition, which plays an important role in quantum information. It is worth to mention that similar characterizations of positive maps defined

on $\mathbb{M}_2(\mathbb{C})$ were considered in [11, 12]. Further, in section 4 we study stability of dynamics of quadratic operators. Note that in [17] we have studied very simple dynamics of quadratic operators. Moreover, we note that the considered quadratic operators are related to quantum groups introduced in [23]. Certain class of quantum groups on $\mathbb{M}_2(\mathbb{C})$ were investigated in [20].

2. Preliminaries

In what follows, by $\mathbb{M}_2(\mathbb{C})$ we denote an algebra of 2×2 matrices over complex filed $\mathbb{C}$. By $\mathbb{M}_2(\mathbb{C})\otimes\mathbb{M}_2(\mathbb{C})$ we mean tensor product of $\mathbb{M}_2(\mathbb{C})$ into itself. We note that such a product can be considered as an algebra of 4×4 matrices $\mathbb{M}_4(\mathbb{C})$ over $\mathbb{C}$. In the sequel $\mathbb{1}$ means an identity matrix, i.e. $\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. By $S(\mathbb{M}_2(\mathbb{C}))$ we denote the set of all states (i.e. linear positive functionals which take value 1 at $\mathbb{1}$) defined on $\mathbb{M}_2(\mathbb{C})$.

Definition 2.1. A linear operator $\Delta : \mathbb{M}_2(\mathbb{C}) \to \mathbb{M}_2(\mathbb{C})\otimes\mathbb{M}_2(\mathbb{C})$ is said to be

(a) – a *quantum quadratic operator (q.q.o.)* if it satisfies the following conditions:
  (i) unital, i.e. $\Delta\mathbb{1} = \mathbb{1}\otimes\mathbb{1}$;
  (ii) Δ is positive, i.e. $\Delta x \geq 0$ whenever $x \geq 0$;

(b) – a *quantum convolution* if it is a q.q.o. and satisfies coassociativity condition:
$$(\Delta\otimes id)\Delta = (id\otimes\Delta)\Delta,$$
where id is the identity operator of $\mathbb{M}_2(\mathbb{C})$;

(c) – a *Kadison-Schwartz operator (KS)* if it satisfies
$$\Delta(x^*x) \geq \Delta(x^*)\Delta(x) \quad \text{for all } x\in\mathbb{M}_2(\mathbb{C}). \tag{2.1}$$

One can see that if Δ is unital and KS operator, then it is a q.q.o. A state $h\in S(\mathbb{M}_2(\mathbb{C}))$ is called *a Haar state* for a q.q.o. Δ if for every $x\in\mathbb{M}_2(\mathbb{C})$ one has
$$(h\otimes id)\circ\Delta(x) = (id\otimes h)\circ\Delta(x) = h(x)\mathbb{1}. \tag{2.2}$$

Remark 2.2. Note that if a quantum convolution Δ on $\mathbb{M}_2(\mathbb{C})$ becomes a $*$-homomorphic map with a condition
$$\overline{\mathrm{Lin}}((\mathbb{1}\otimes\mathbb{M}_2(\mathbb{C}))\Delta(\mathbb{M}_2(\mathbb{C}))) = \overline{\mathrm{Lin}}((\mathbb{M}_2(\mathbb{C})\otimes\mathbb{1})\Delta(\mathbb{M}_2(\mathbb{C}))) = \mathbb{M}_2(\mathbb{C})\otimes\mathbb{M}_2(\mathbb{C})$$
then a pair $(\mathbb{M}_2(\mathbb{C}),\Delta)$ is called a *compact quantum group* [23, 20]. It is known [23] that for given any compact quantum group there exists a unique Haar state for Δ.

Remark 2.3. Let $U : \mathbb{M}_2(\mathbb{C})\otimes\mathbb{M}_2(\mathbb{C}) \to \mathbb{M}_2(\mathbb{C})\otimes\mathbb{M}_2(\mathbb{C})$ be a linear operator such that $U(x\otimes y) = y\otimes x$ for all $x, y\in\mathbb{M}_2(\mathbb{C})$. If a q.q.o. Δ satisfies $U\Delta = \Delta$, then Δ is called a *quantum quadratic stochastic operator*. Such a kind of operators were studied and investigated in [15, 17].

Each q.q.o. Δ defines a conjugate operator $\Delta^* : (\mathbb{M}_2(\mathbb{C})\otimes\mathbb{M}_2(\mathbb{C}))^* \to \mathbb{M}_2(\mathbb{C})^*$ by
$$\Delta^*(f)(x) = f(\Delta x), \quad f\in(\mathbb{M}_2(\mathbb{C})\otimes\mathbb{M}_2(\mathbb{C}))^*, \ x\in\mathbb{M}_2(\mathbb{C}). \tag{2.3}$$

One can define an operator V_Δ by
$$V_\Delta(\varphi) = \Delta^*(\varphi\otimes\varphi), \quad \varphi\in S(\mathbb{M}_2(\mathbb{C})), \tag{2.4}$$
which is called a *quadratic operator (q.o.)*. Thanks to the conditions (i),(ii) of Def. 2.1 the operator V_Δ maps $S(\mathbb{M}_2(\mathbb{C}))$ to $S(\mathbb{M}_2(\mathbb{C}))$. In some literature operator V_Δ is called quadratic convolution (see for example [6]).

3. Quantum quadratic operators on $\mathbb{M}_2(\mathbb{C})$

In this section we are going to describe quantum quadratic operators $\mathbb{M}_2(\mathbb{C})$ as well as find necessary conditions for such operators to satisfy the Kadison-Schwartz property.

Recall [3] that the identity and Pauli matrices $\{\mathbb{1}, \sigma_1, \sigma_2, \sigma_3\}$ form a basis for $\mathbb{M}_2(\mathbb{C})$, where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In this basis every matrix $x \in \mathbb{M}_2(\mathbb{C})$ can be written as $x = w_0\mathbb{1} + \mathbf{w}\sigma$ with $w_0 \in \mathbb{C}$, $\mathbf{w} = (w_1, w_2, w_3) \in \mathbb{C}^3$, here $\mathbf{w}\sigma = w_1\sigma_1 + w_2\sigma_2 + w_3\sigma_3$.

Lemma 3.1. [19] *The following assertions hold true:*

(a) x *is self-adjoint iff* $w_0, \mathbf{w}$ *are reals;*
(b) $\mathrm{Tr}(x) = 1$ *iff* $w_0 = 0.5$, *here* Tr *is the trace of a matrix* x*;*
(c) $x > 0$ *iff* $\|\mathbf{w}\| \le w_0$, *where* $\|\mathbf{w}\| = \sqrt{|w_1|^2 + |w_2|^2 + |w_3|^2}$.

As well as any state $\varphi \in S(\mathbb{M}_2(\mathbb{C}))$ can be represented by

$$\varphi(w_0\mathbb{1} + \mathbf{w}\sigma) = w_0 + \langle \mathbf{w}, \mathbf{f}\rangle, \tag{3.1}$$

where $\mathbf{f} = (f_1, f_2, f_3) \in \mathbb{R}^3$ such that $\|\mathbf{f}\| \le 1$. Here as before $\langle\cdot,\cdot\rangle$ stands for the scalar product in $\mathbb{C}^3$. Therefore, in the sequel we will identify a state with a vector $\mathbf{f} \in \mathbb{R}^3$.

In what follows by τ we denote a normalized trace,

$$\tau\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \frac{x_{11} + x_{22}}{2},$$

i.e. $\tau(x) = \frac{1}{2}\mathrm{Tr}(x)$, $x \in \mathbb{M}_2(\mathbb{C})$,

Let $\Delta : \mathbb{M}_2(\mathbb{C}) \to \mathbb{M}_2(\mathbb{C}) \otimes \mathbb{M}_2(\mathbb{C})$ be a q.q.o. with a Haar state τ. Then one has

$$\tau \otimes \tau(\Delta x) = \tau(\tau \otimes id)(\Delta(x)) = \tau(x)\tau(\mathbb{1}) = \tau(x), \quad x \in \mathbb{M}_2(\mathbb{C}),$$

which means that τ is an invariant state for Δ.

Let us write the operator Δ in terms of a basis in $\mathbb{M}_2(\mathbb{C}) \otimes \mathbb{M}_2(\mathbb{C})$ formed by the Pauli matrices. Namely,

$$\Delta\mathbb{1} = \mathbb{1} \otimes \mathbb{1};$$

$$\Delta(\sigma_i) = b_i(\mathbb{1} \otimes \mathbb{1}) + \sum_{j=1}^{3} b_{ji}^{(1)}(\mathbb{1} \otimes \sigma_j) + \sum_{j=1}^{3} b_{ji}^{(2)}(\sigma_j \otimes \mathbb{1}) + \sum_{m,l=1}^{3} b_{ml,i}(\sigma_m \otimes \sigma_l),$$

where $i = 1, 2, 3$.

Now we want to describe all q.q.o. which has a Haar state τ.

Note that the positivity of Δ implies that $\Delta x^* = (\Delta x)^*$, therefore we have

$$\Delta(\sigma_i^*) = \overline{b_i}(\mathbb{1} \otimes \mathbb{1}) + \sum_{j=1}^{3} \overline{b_{ji}^{(1)}}(\mathbb{1} \otimes \sigma_j) + \sum_{j=1}^{3} \overline{b_{ji}^{(2)}}(\sigma_j \otimes \mathbb{1}) + \sum_{m,l=1}^{3} \overline{b_{ml,i}}(\sigma_m \otimes \sigma_l).$$

This yields that $b_i = \overline{b_i}$, $b_{ji}^{(k)} = \overline{b_{ji}^{(k)}}$ $(k = 1, 2)$ and $b_{ml,i} = \overline{b_{ml,i}}$, i.e. all coefficients are real numbers.

Now the invariance of Δ w.r.t. τ means $\tau \otimes \tau(\Delta(\sigma_i)) = \tau(\sigma_i) = 0$ which implies that $b_j = 0$, $j = 1, 2, 3$.

The equality (2.2) with $h=\tau$ yields that

$$\begin{aligned}(id\otimes\tau)\Delta(\sigma_i) &= (id\otimes\tau)\Big[\sum_{j=1}^3\big(b_{ji}^{(1)}(\mathbb{1}\otimes\sigma_j)+b_{ji}^{(2)}(\sigma_j\otimes\mathbb{1})\big)+\sum_{m,l=1}^3 b_{ml,i}(\sigma_m\otimes\sigma_l)\Big]\\ &= \sum_{j=1}^3 b_{ji}^{(2)}\sigma_j=\tau(\sigma_j)\mathbb{1}=0.\end{aligned}$$

Therefore, $b_{ji}^{(2)}=0$, for all $i,j=1,2,3$. Now similarly, from

$$(\tau\otimes id)\Delta(\sigma_j)=\sum_{j=1}^3 b_{ji}^{(1)}\sigma_j=\tau(\sigma_j)\mathbb{1},$$

one gets $b_{ji}^{(1)}=0$. Hence, Δ has the following form

$$\Delta(\sigma_i)=\sum_{m,l=1}^3 b_{ml,i}(\sigma_m\otimes\sigma_l),\quad i=1,2,3. \tag{3.2}$$

Let us turn to the positivity of Δ. Given vector $\mathbf{f}=(f_1,f_2,f_3)\in\mathbb{R}^3$ put

$$\beta(\mathbf{f})_{ij}=\sum_{k=1}^3 b_{ki,j}f_k. \tag{3.3}$$

Define a matrix $\mathbb{B}(\mathbf{f})=(\beta(\mathbf{f})_{ij})_{ij=1}^3$.

Now given a state φ, (i.e. $\varphi(x)=\omega_0+\langle\mathbf{w},\mathbf{f}\rangle$, $\mathbf{f}\in\mathbb{R}^3$,$\|\mathbf{f}\|\leq1$) by E_φ we denote the canonical conditional expectation defined by $E_\varphi(x\otimes y)=\varphi(x)y$, where $x,y\in\mathbb{M}_2(\mathbb{C})$.

By $\|\mathbb{B}(\mathbf{f})\|$ we denote a norm of the matrix $\mathbb{B}(\mathbf{f})$ associated with Euclidean norm in $\mathbb{C}^3$. Put

$$S=\{\mathbf{p}=(p_1,p_2,p_3)\in\mathbb{R}^3:\ p_1^2+p_2^2+p_3^2\leq1\}$$

and denote

$$|||\mathbb{B}|||=\sup_{\mathbf{f}\in S}\|\mathbb{B}(\mathbf{f})\|.$$

Proposition 3.2. *Let Δ be a q.q.o. with a Haar state τ, then $|||\mathbb{B}|||\leq1$.*

Proof. Let $x\in\mathbb{M}_2(\mathbb{C})$ be a positive element. Note that for any $x=w_0\mathbb{1}+\mathbf{w}\sigma\in\mathbb{M}_2(\mathbb{C})$ from (3.2) one finds

$$\Delta(x)=w_0\mathbb{1}\otimes\mathbb{1}+\sum_{m,l=1}^3\langle\mathbf{b}_{ml},\overline{\mathbf{w}}\rangle\sigma_m\otimes\sigma_u, \tag{3.4}$$

here we have denoted we $\mathbf{b}_{ml}=(b_{ml,1},b_{ml,2},b_{ml,3})$ and as before $\langle\cdot,\cdot\rangle$ stands for the scalar product in $\mathbb{C}^3$.

Therefore, we have

$$\begin{aligned}E_\varphi(\Delta(x)) &= w_0\mathbb{1}+\sum_{i,j=1}^3\langle\mathbf{b}_{i,j},\overline{\mathbf{w}}\rangle f_i\sigma_j\\ &= w_0\mathbb{1}+\mathbb{B}(\mathbf{f})\mathbf{w}\sigma\end{aligned}$$

where $\varphi(x) = w_0 + \langle \mathbf{f}, \mathbf{w}\rangle$, $\mathbf{f} = (f_1, f_2, f_3) \in S$, and we have used $\varphi(\sigma_i) = f_i$ and

$$\begin{aligned} \sum_{i=1}^{3} \langle \mathbf{b}_{i,j}, \overline{\mathbf{w}}\rangle f_i &= \sum_{l=1}^{3}\sum_{i=1}^{3} b_{ij,l} f_i w_l \\ &= \sum_{l=1}^{3} \beta_{jl}(\mathbf{f}) w_l \\ &= (\mathbb{B}(\mathbf{f})\mathbf{w})_j \end{aligned}$$

Positivity of x yields that $E_\varphi(\Delta(x))$ is positive, for all states φ, since E_φ is a conditional expectation. Hence, according to Lemma 3.1 positivity of $E_\varphi(\Delta(x))$ equivalent to $\|\mathbb{B}(\mathbf{f})\mathbf{w}\| \leq w_0$ for all $\mathbf{f}$ and $\mathbf{w}$ with $\|\mathbf{w}\| < w_0$. Consequently, one finds that $\|\mathbb{B}(\mathbf{f})\| = \sup\limits_{\|\mathbf{w}\|\leq 1} \|\mathbb{B}(\mathbf{f})\mathbf{w}\| \leq 1$, which yields the assertion. □

Now we are ready to formulate a description result for q.q.o.

Theorem 3.3. *Let $\Delta : \mathbb{M}_2(\mathbb{C}) \to \mathbb{M}_2(\mathbb{C}) \otimes \mathbb{M}_2(\mathbb{C})$ be a q.q.o. with a Haar state τ, then it has the following form:*

$$\Delta(x) = w_0 \mathbb{1} \otimes \mathbb{1} + \sum_{m,l=1}^{3} \langle \mathbf{b}_{ml}, \overline{\mathbf{w}}\rangle \sigma_m \otimes \sigma_l, \tag{3.5}$$

where $x = w_0 + \mathbf{w}\sigma$, $\mathbf{b}_{ml} = (b_{ml,1}, b_{ml,2}, b_{ml,3})$ and the coefficients $\{b_{ml,k}\}$ satisfy $\||\mathbb{B}\|| \leq 1$.

Remark 3.4. Note that similar characterizations of positive maps defined on $\mathbb{M}_2(\mathbb{C})$ were considered in [12] (see also [8]). Characterization of completely positive mappings from $\mathbb{M}_2(\mathbb{C})$ into itself with invariant state τ was established in [19].

Next we would like to find some conditions for q.q.o. to be Kadison-Schwartz operators. To do it, we need some auxiliary results.

Lemma 3.5. *Let $\mathbf{a}, \mathbf{c} \in \mathbb{C}^3$. Then one has*

$$(\mathbf{a}\sigma)\cdot(\overline{\mathbf{c}}\sigma) - (\mathbf{c}\sigma)\cdot(\overline{\mathbf{a}}\sigma) = \big(\langle \mathbf{a}, \mathbf{c}\rangle - \langle \mathbf{c}, \mathbf{a}\rangle\big)\mathbb{1} + i\big([\mathbf{a}, \overline{\mathbf{c}}] + [\overline{\mathbf{a}}, \mathbf{c}]\big)\sigma \tag{3.6}$$

$$(\mathbf{a}\sigma)\cdot(\overline{\mathbf{a}}\sigma) = \|\mathbf{a}\|^2 \mathbb{1} + i[\mathbf{a}, \overline{\mathbf{a}}]\sigma \tag{3.7}$$

The proof is straightforward.

Let $\Delta : \mathbb{M}_2(\mathbb{C}) \to \mathbb{M}_2(\mathbb{C}) \otimes \mathbb{M}_2(\mathbb{C})$ be a linear operator with a Haar state τ, then it has a form (3.5). We would like to find some conditions to the coefficients $\{b_{ml,k}\}$ when Δ is a Kadison-Schwartz operator. Given $x = w_0 + \mathbf{w}\sigma$ denoting $x_{ml} = \langle \mathbf{b}_{ml}, \mathbf{w}\rangle$ one has

$$\begin{aligned} \Delta(x^*x) &= (|w_0|^2 + \|\mathbf{w}\|^2)\mathbb{1} + \sum_{m,l=1}^{3} (\overline{w}_0 \overline{x}_{m,l} + w_0 x_{m,l})\sigma_m \otimes \sigma_l \\ &\quad + i \sum_{m,l=1}^{3} \langle \mathbf{b}_{m,l}, [\mathbf{w}, \overline{\mathbf{w}}]\rangle \sigma_m \otimes \sigma_l \end{aligned} \tag{3.8}$$

$$\begin{aligned}\Delta(x)^*\Delta(x) &= |w_0|^2\mathbb{1} + \sum_{m,l=1}^{3}(\overline{w}_0\overline{x}_{m,l} + w_0 x_{m,l})\sigma_m\otimes\sigma_l \\ &\quad + \Big(\sum_{m,l=1}^{3} x_{m,l}\sigma_m\otimes\sigma_l\Big)\Big(\sum_{m,l=1}^{3}\overline{x_{m,l}}\sigma_m\otimes\sigma_l\Big). \end{aligned} \tag{3.9}$$

Putting $\mathbf{x}_m = (x_{m1}, x_{m2}, x_{m3})$, $m = 1, 2, 3$ and using this notation we rewrite the last term of the equality (3.9) as follows

$$\begin{aligned}\Big(\sum_{m,l=1}^{3} x_{m,l}\sigma_m\otimes\sigma_l\Big)\Big(\sum_{m,l=1}^{3}\overline{x_{m,l}}\sigma_m\otimes\sigma_l\Big) &= \Big(\sum_{m=1}^{3}\sigma_m\otimes(\mathbf{x}_m\sigma)\Big)\Big(\sum_{m=1}^{3}\sigma_m\otimes(\overline{\mathbf{x}}_m\sigma)\Big)\\ &= \mathbb{1}\otimes\sum_{k=1}^{3}(\mathbf{x}_k\sigma)\cdot(\overline{\mathbf{x}}_k\sigma)\\ &\quad + i\sigma_1\otimes\big((\mathbf{x}_2\sigma)\cdot(\overline{\mathbf{x}}_3\sigma) - (\mathbf{x}_3\sigma)\cdot(\overline{\mathbf{x}}_2\sigma)\big)\\ &\quad + i\sigma_2\otimes\big((\mathbf{x}_3\sigma)\cdot(\overline{\mathbf{x}}_1\sigma) - (\mathbf{x}_1\sigma)\cdot(\overline{\mathbf{x}}_3\sigma)\big)\\ &\quad + i\sigma_3\otimes\big((\mathbf{x}_1\sigma)\cdot(\overline{\mathbf{x}}_2\sigma) - (\mathbf{x}_2\sigma)\cdot(\overline{\mathbf{x}}_1\sigma)\big)\end{aligned}$$

According to Lemma 3.5 the last equality equal to

$$\begin{aligned} X := \quad & \mathbb{1}\otimes\Big(\sum_{j=1}^{3}\|\mathbf{x}_j\|^2\mathbb{1} + i\sum_{j=1}^{3}[\mathbf{x}_j, \overline{\mathbf{x}}_j]\sigma)\Big)\\ & + i\sigma_1\otimes\Big((\langle\mathbf{x}_2, \mathbf{x}_3\rangle - \langle\mathbf{x}_3, \mathbf{x}_2\rangle)\mathbb{1} + i([\mathbf{x}_2, \overline{\mathbf{x}_3}] + [\overline{\mathbf{x}_2}, \mathbf{x}_3])\sigma\Big)]\\ & + i\sigma_2\otimes\Big((\langle\mathbf{x}_3, \mathbf{x}_1\rangle - \langle\mathbf{x}_1, \mathbf{x}_3\rangle)\mathbb{1} + i([\mathbf{x}_3, \overline{\mathbf{x}_1}] + [\overline{\mathbf{x}_3}, \mathbf{x}_1])\sigma\Big)\\ & + i\sigma_3\otimes\Big((\langle\mathbf{x}_1, \mathbf{x}_2\rangle - \langle\mathbf{x}_2, \mathbf{x}_1\rangle)\mathbb{1} + i([\mathbf{x}_1, \overline{\mathbf{x}_2}] + [\overline{\mathbf{x}_1}, \mathbf{x}_2])\sigma\Big)\end{aligned}$$

Then from (3.8), (3.9) one gets that

$$\Delta(x^*x) - \Delta(x)^*\Delta(x) = \|\mathbf{w}\|^2\mathbb{1} + \sum_{m,l=1}^{3}\langle\mathbf{b}_{ml}, [\mathbf{w}, \overline{\mathbf{w}}]\rangle\sigma_m\otimes\sigma_l - X.$$

Now take an arbitrary state $\varphi\in S(\mathbb{M}_2(\mathbb{C}))$ and consider

$$E_\varphi(\Delta(x^*x) - \Delta(x)^*\Delta(x)) = \|w\|^2\mathbb{1} + i\sum_{m,l=1}^{3}\langle\mathbf{b}_{ml}, [\mathbf{w}, \overline{\mathbf{w}}]\rangle f_m\sigma_l - E_\varphi(X), \tag{3.10}$$

where $\varphi(\sigma_m) = f_m$.

Denote

$$\alpha_{ml} = \langle\mathbf{x}_m, \mathbf{x}_l\rangle - \langle\mathbf{x}_l, \mathbf{x}_m\rangle, \quad \gamma_{ml} = [\mathbf{x}_m, \overline{\mathbf{x}_l}] + [\overline{\mathbf{x}_m}, \mathbf{x}_l]. \tag{3.11}$$

By π we shall denote mapping $\{1, 2, 3, 4\}$ to $\{1, 2, 3\}$ defined by $\pi(1) = 2, \pi(2) = 3, \pi(3) = 1, \pi(4) = \pi(1)$. Then one can find

$$
\begin{aligned}
E_{\varphi}(X) &= \sum_{m=1}^{3}\|\mathbf{x}_m\|^2\mathbb{1} + i\sum_{m=1}^{3}[\mathbf{x}_m,\overline{\mathbf{x}}_m]\sigma \\
&\quad + i\sum_{m=1}^{3} f_m(\alpha_{\pi(m),\pi(m+1)}\mathbb{1} + i\gamma_{\pi(m),\pi(m+1)}\sigma) \qquad (3.12)
\end{aligned}
$$

Denoting

$$
\mathbf{q}(\mathbf{f},\mathbf{w}) = \big(\langle\beta(\mathbf{f})_1,[\mathbf{w},\overline{\mathbf{w}}]\rangle, \langle\beta(\mathbf{f})_2,[\mathbf{w},\overline{\mathbf{w}}]\rangle, \langle\beta(\mathbf{f})_3,[\mathbf{w},\overline{\mathbf{w}}]\rangle\big), \tag{3.13}
$$

where $\beta(\mathbf{f})_m = \big(\beta(\mathbf{f})_{m1},\beta(\mathbf{f})_{m2},\beta(\mathbf{f})_{m3}\big)$ (see (3.3)), and substituting the last equality (3.12) to (3.10) we obtain

$$
\begin{aligned}
E_{\varphi}(\Delta(x^*x)-\Delta(x)^*\Delta(x)) &= \left(\|\mathbf{w}\|^2 - i\sum_{m=1}^{3} f_m\alpha_{\pi(m),\pi(m+1)} - \sum_{m=1}^{3}\|\mathbf{x}_m\|^2\right)\mathbb{1} \\
&\quad + i\left(\mathbf{q}(\mathbf{f},\mathbf{w}) - i\sum_{m=1}^{3} f_m\gamma_{\pi(m),\pi(m+1)} - [\mathbf{x}_m,\overline{\mathbf{x}}_m]\right)\sigma. \qquad (3.14)
\end{aligned}
$$

So, thanks to Lemma 3.1 the right hand side of (3.14) is positive if and only if

$$
\|\mathbf{w}\|^2 - i\sum_{m=1}^{3} f_m\alpha_{\pi(m),\pi(m+1)} - \sum_{m=1}^{3}\|\mathbf{x}_m\|^2 \geq 0 \tag{3.15}
$$

$$
\left\|\mathbf{q}(\mathbf{f},\mathbf{w}) - i\sum_{m=1}^{3} f_m\gamma_{\pi(m),\pi(m+1)} - [\mathbf{x}_m,\overline{\mathbf{x}}_m]\right\| \leq \|\mathbf{w}\|^2 - i\sum_{k=1}^{3} f_k\alpha_{\pi(k),\pi(k+1)} - \sum_{m=1}^{3}\|\mathbf{x}_m\|^2. \tag{3.16}
$$

Note that here the numbers α_{ml} are skew-symmetric, i.e. $\overline{\alpha_{ml}} = -\alpha_{ml}$, therefore, the equality (3.17) has a sense.

According to Theorem 3.3 we obtain the following

Theorem 3.6. *Let $\Delta : \mathbb{M}_2(\mathbb{C}) \to \mathbb{M}_2(\mathbb{C})\otimes\mathbb{M}_2(\mathbb{C})$ be a Kadison-Schwartz operator with a Haar state τ, then it has the form* (3.5) *and the coefficients $\{b_{ml,k}\}$ satisfy conditions*

$$
\|\mathbf{w}\|^2 - i\sum_{m=1}^{3} f_m\alpha_{\pi(m),\pi(m+1)} - \sum_{m=1}^{3}\|\mathbf{x}_m\|^2 \geq 0
$$

$$
\left\|\mathbf{q}(\mathbf{f},\mathbf{w}) - i\sum_{m=1}^{3} f_m\gamma_{\pi(m),\pi(m+1)} - [\mathbf{x}_m,\overline{\mathbf{x}}_m]\right\| \leq \|\mathbf{w}\|^2 - i\sum_{k=1}^{3} f_k\alpha_{\pi(k),\pi(k+1)} - \sum_{m=1}^{3}\|\mathbf{x}_m\|^2.
$$

for all $\mathbf{f}\in S, \mathbf{w}\in\mathbb{C}^3$. Here as before $\mathbf{x}_m = \big(\langle\mathbf{b}_{m1},\mathbf{w}\rangle,\langle\mathbf{b}_{m2},\mathbf{w}\rangle,\langle\mathbf{b}_{m3},\mathbf{w}\rangle\big)$, $\mathbf{b}_{ml} = (b_{ml,1},b_{ml,2},b_{ml,3})$ and $\mathbf{q}(\mathbf{f},\mathbf{w})$, α_{ml} and γ_{ml} are defined in (3.13),(3.11)*, respectively.*

Corollary 3.7. *Let $\Delta : \mathbb{M}_2(\mathbb{C}) \to \mathbb{M}_2(\mathbb{C}) \otimes \mathbb{M}_2(\mathbb{C})$ be a Kadison-Schwartz operator given by* (3.5)*, then the coefficients $\{b_{ml,k}\}$ satisfy the following conditions*

$$\sum_{m=1}^{3} \|\mathbf{x}_m\|^2 + i\alpha_{2,3} \leq \|\mathbf{w}\|^2, \tag{3.17}$$

$$\left\| \boldsymbol{h}(\mathbf{w}) - i\gamma_{2,3} + i\sum_{m=1}^{3} [\mathbf{x}_m, \overline{\mathbf{x}}_m] \right\| \leq \|\mathbf{w}\|^2 - i\alpha_{2,3} - \sum_{m=1}^{3} \|\mathbf{x}_m\|^2, \tag{3.18}$$

where $\boldsymbol{h}(\mathbf{w}) = \big(\langle \mathbf{b}_{11}, [\mathbf{w}, \overline{\mathbf{w}}]\rangle, \langle \mathbf{b}_{12}, [\mathbf{w}, \overline{\mathbf{w}}]\rangle, \langle \mathbf{b}_{13}, [\mathbf{w}, \overline{\mathbf{w}}]\rangle\big)$, for all $\mathbf{w} \in \mathbb{C}^3$.

The proof immediately follows from the previous Theorem 3.6 when we take $\mathbf{f} = (1, 0, 0)$ in (3.17),(3.17).

Remark 3.8. The provided characterization with [13] allows us to construct examples of positive or Kadison-Schwartz operators which are not completely positive (see subsection 4.3).

4. Dynamics of quantum quadratic operators

4.1. **General case.** In this section we are going to study dynamics of convolution operator V_Δ associated with a q.q.o. Δ defined on $\mathbb{M}_2(\mathbb{C})$.

Let $\Delta : \mathbb{M}_2(\mathbb{C}) \to \mathbb{M}_2(\mathbb{C}) \otimes \mathbb{M}_2(\mathbb{C})$ be a liner operator with a Haar state τ. Then due to Theorem 3.3 Δ has a form (3.5). Take arbitrary states $\varphi, \psi \in S(\mathbb{M}_2(\mathbb{C}))$ and $\mathbf{f}, \mathbf{p} \in S$ be the corresponding vectors (see (3.1)). Then one finds that

$$\Delta^*(\varphi \otimes \psi)(\sigma_k) = \sum_{i,j=1}^{3} b_{ij,k} f_i p_j, \quad k = 1, 2, 3.$$

Thanks to Lemma 3.1 the functional $\Delta^*(\varphi \otimes \psi)$ is a state if and only if the vector

$$\mathbf{f}_{\Delta^*(\varphi,\psi)} = \left(\sum_{i,j=1}^{3} b_{ij,1} f_i p_j, \sum_{i,j=1}^{3} b_{ij,2} f_i p_j, \sum_{i,j=1}^{3} b_{ij,3} f_i p_j \right).$$

satisfies $\|\mathbf{f}_{\Delta^*(\varphi,\psi)}\| \leq 1$.

So, we have the following

Proposition 4.1. *Let $\Delta : \mathbb{M}_2(\mathbb{C}) \to \mathbb{M}_2(\mathbb{C}) \otimes \mathbb{M}_2(\mathbb{C})$ be a liner operator with a Haar state τ. Then $\Delta^*(\cdot \otimes \cdot)$ bilinear form is positive if and only if one holds*

$$\sum_{k=1}^{3} \left| \sum_{i,j=1}^{3} b_{ij,k} f_i p_j \right|^2 \leq 1 \quad \text{for all} \;\; \mathbf{f}, \mathbf{p} \in S. \tag{4.1}$$

From the proof of Proposition 3.2 and the last proposition we get

Corollary 4.2. *Let $\mathbb{B}(\mathbf{f})$ be the corresponding matrix to an operator given by* (3.5)*. Then $|||\mathbb{B}||| \leq 1$ if and only if* (4.1) *is satisfied.*

Let us find some sufficient condition for the coefficients $\{b_{ij,k}\}$ to satisfy (4.1).

Corollary 4.3. *Let*

$$\sum_{i,j,k=1}^{3} |b_{ij,k}|^2 \leq 1 \tag{4.2}$$

be satisfied, then (4.1) *holds.*

Proof. Let (4.2) be satisfied. Take any $\mathbf{f}, \mathbf{p} \in S$, then

$$\begin{aligned}\left|\sum_{i,j=1}^{3} b_{ij,k} f_i p_j\right|^2 &\leq \left(\sum_{i,j=1}^{3} |b_{ij,k}||f_i p_j|\right)^2 \\ &\leq \sum_{i,j=1}^{3} |b_{ij,k}|^2 \sum_{i=1}^{3} |f_i|^2 \sum_{i,j=1}^{3} |p_j|^2 \\ &\leq \sum_{i,j=1}^{3} |b_{ij,k}|^2\end{aligned}$$

which implies the assertion. □

Let us consider the quadratic operator, which is defined as $V_\Delta(\varphi) = \Delta^*(\varphi \otimes \varphi)$, $\varphi \in S(\mathbb{M}_2(\mathbb{C}))$. From Theorem 3.3 and Corollary 4.2 one can see that the defined operator V_Δ maps $S(\mathbb{M}_2(\mathbb{C}))$ into itself if and only if $|||\mathbb{B}||| \leq 1$. From (3.5) we find that

$$V_\Delta(\varphi)(\sigma_k) = \sum_{i,j=1}^{3} b_{ij,k} f_i f_j, \quad \mathbf{f} \in S.$$

This suggests us to consider a nonlinear operator $V : S \to S$ defined by

$$V(\mathbf{f})_k = \sum_{i,j=1}^{3} b_{ij,k} f_i f_j, \quad k = 1, 2, 3. \tag{4.3}$$

where $\mathbf{f} = (f_1, f_2, f_3) \in S$. Furthermore, we are going to study dynamics of V.

Since S is a convex compact set, then due to Schauder theorem V has at least one fixed point. One can see that one of the fixed points is $(0, 0, 0)$. Furthermore, we will be interested on uniqueness (stability) of this fixed point.

Let us take $\mathbf{f}, \mathbf{p} \in S$ and consider the difference

$$\begin{aligned}|V(\mathbf{f})_k - V(\mathbf{p})_k| &\leq \sum_{i,j=1}^{3} |b_{ij,k}||f_i f_j - p_i p_j| \\ &\leq \sum_{i,j=1}^{3} |b_{ij,k}||f_i||f_j - p_j| + \sum_{i,j=1}^{3} |b_{ij,k}||p_j||f_i - p_i| \\ &\leq \sum_{i,j=1}^{3} |b_{ij,k}||f_j - p_j| + \sum_{i,j=1}^{3} |b_{ij,k}||f_i - p_i| \\ &\leq \left(\sqrt{\sum_{j=1}^{3}\left(\sum_{i=1}^{3} |b_{ij,k}|\right)^2} + \sqrt{\sum_{i=1}^{3}\left(\sum_{j=1}^{3} |b_{ij,k}|\right)^2}\right)\|\mathbf{f} - \mathbf{p}\|.\end{aligned}$$

where $k = 1, 2, 3$.

Therefore, one can calculate that $\|V(\mathbf{f}) - V(\mathbf{g})\| \leq \sqrt{\alpha}\|\mathbf{f} - \mathbf{p}\|$, where

$$\alpha = \sum_{k=1}^{3}\left(\sqrt{\sum_{j=1}^{3}\left(\sum_{i=1}^{3} |b_{ij,k}|\right)^2} + \sqrt{\sum_{i=1}^{3}\left(\sum_{j=1}^{3} |b_{ij,k}|\right)^2}\right)^2 \tag{4.4}$$

Theorem 4.4. *If $\alpha < 1$ then V is a contraction, hence $(0, 0, 0)$ is a unique stable fixed point.*

Let us investigate more weaker conditions than given in Theorem 4.4.

Take any $\mathbf{f} \in S$ and denote $\gamma_{\mathbf{f}} = \max\{|f_1|, |f_2|, |f_3|\}$. It is clear that $\gamma_{\mathbf{f}} \leq 1$. Put

$$\delta_k = \sum_{i,j=1}^{3} |b_{ij,k}|, \quad k = 1,2,3. \tag{4.5}$$

Define a new operator $\tilde{V} : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$\tilde{V}(\mathbf{p})_k = \sum_{i,j=1}^{3} |b_{ij,k}| p_i p_j, \quad \mathbf{p} \in \mathbb{R}^3, \ k = 1,2,3. \tag{4.6}$$

Then one can see that

$$|V(\mathbf{f})_k\| \leq \gamma_{\mathbf{f}}^2 \sum_{i,j=1}^{3} |b_{ij,k}| = \gamma_{\mathbf{f}}^2 \delta_k.$$

Similarly, we find

$$|V^2(\mathbf{f})_k| \leq \sum_{i,j=1}^{3} |b_{ij,k}||V(\mathbf{f})_i||V(\mathbf{f})_j| \leq \gamma_{\mathbf{f}}^{2^2} \tilde{V}(\mathbf{d})_k, \quad k = 1,2,3,$$

where $\mathbf{d} = (\delta_1, \delta_2, \delta_3)$.

Hence, using mathematical induction one can get

$$|V^n(\mathbf{f})_k| \leq \gamma_{\mathbf{f}}^{2^n} \tilde{V}^{n-1}(\mathbf{d})_k, \quad n \in \mathbb{N}. \tag{4.7}$$

So, if $\gamma_{\mathbf{f}} < 1$ and the sequence $\{\tilde{V}^n(\mathbf{d})_k\}$ is bounded, then (4.7) implies that $V(\mathbf{f})_k \to 0$ as $n \to \infty$.

Lemma 4.5. *Assume that one has*

$$\sum_{i,j=1}^{3} |b_{ij,k}| \leq 1, \quad k = 1,2,3. \tag{4.8}$$

Then the sequence $\{\tilde{V}^n(\mathbf{d})_k\}$ is bounded.

Proof. From (4.8) we conclude that $\delta_k \leq 1$ for every $k = 1,2,3$. Therefore, it follows from (4.6) that

$$|\tilde{V}(\mathbf{d})_k| = \sum_{i,j=1}^{3} |b_{ij,k}| \delta_i \delta_j \leq \delta_k \leq 1$$

which yields that $|\tilde{V}^n(\mathbf{d})_k| \leq 1$ for every $n \in \mathbb{N}$, $k = 1,2,3$. □

Now we are interested when the sequence $\{\tilde{V}^n(\mathbf{d})\}$ converges to 0.

Lemma 4.6. *Assume that (4.8) is satisfied. Then following assertions hold true:*

(i) *If there is $n_0 \in \mathbb{N}$ such that $\tilde{V}^{n_0}(\mathbf{d})_k < 1$ for every $k = 1,2,3$, then $\tilde{V}^n(\mathbf{d}) \to 0$ as $n \to \infty$;*

(ii) *If there is $k_0 \in \{1,2,3\}$ such that $\delta_{k_0} < 1$ and for every $k = 1,2,3$ one can find $i_0 \in \{1,2,3\}$ with $|b_{i_0,k_0,k}| + |b_{k_0,i_0,k}| \neq 0$, then $\tilde{V}^n(\mathbf{d}) \to 0$ as $n \to \infty$.*

Proof. (i) Let us denote $v = \max\{V^{n_0}(\mathbf{d})_1, V^{n_0}(\mathbf{d})_k, V^{n_0}(\mathbf{d})_3\}$. It is clear that $0 < v < 1$. Then from (4.6) one gets

$$\tilde{V}^{n_0+1}(\mathbf{d})_k = \sum_{i,j=1}^{3} |b_{ij,k}| V^{n_0}(\mathbf{d})_i V^{n_0}(\mathbf{d})_j \leq v^2 \delta_k \leq v^2.$$

Iterating this procedure we obtain $\tilde{V}^{n+n_0}(\mathbf{d})_k \leq v^{2^n}$ for every $n \in \mathbb{N}$, $k = 1, 2, 3$. This yields the assertion.

(ii) Take any $k \in \{1, 2, 3\}$. Due to (ii) one can find i_0 such that $|b_{i_0,k_0,k}| + |b_{k_0,i_0,k}| \neq 0$. Then from (4.6) with (4.8) we have

$$\begin{aligned}
V(\mathbf{d})_k &= \sum_{i,j=1}^{3} |b_{ij,k}| \delta_j \delta_j \\
&= \sum_{j=1}^{3} |b_{k_0 j,k}| \delta_{k_0} \delta_j + \sum_{i=1}^{3} |b_{ik_0,k}| \delta_i \delta_{k_0} + \sum_{\substack{i,j=1 \\ i,j \neq k_0}}^{3} |b_{ij,k}| \delta_i \delta_j \\
&\leq \sum_{j=1}^{3} |b_{k_0 j,k}| \delta_{k_0} + \sum_{i=1}^{3} |b_{ik_0,k}| \delta_{k_0} + \sum_{\substack{i,j=1 \\ i,j \neq k_0}}^{3} |b_{ij,k}| \\
&= \delta_k - (1 - \delta_{k_0}) \sum_{i=1}^{3} (|b_{k_0 j,k}| + |b_{jk_0,k}|) < \delta_k \leq 1,
\end{aligned}$$

which due to (i) implies the required assertion. $\square$

So, we have the following

Theorem 4.7. *Let V be a quadratic operator (4.3) associated with a q.q.o. Δ and $\mathbf{f} \in S$. If one of the following conditions hold*

(i) *If $\gamma_{\mathbf{f}} < 1$ (which means that at least two coordinates of $\mathbf{f}$ are not zero) and a sequence $\{\tilde{V}(\mathbf{d})\}$ is bounded, where $\mathbf{d} = (\delta_1, \delta_2, \delta_3)$ and δ_k is given by (4.5).*
(ii) *If $\gamma_{\mathbf{f}} = 1$ and $\tilde{V}(\mathbf{d}) \to 0$ as $n \to \infty$;*
(iii) *If $\gamma_{\mathbf{f}} < 1$ and (4.8) is satisfied;*
(iv) *Let (4.8) be satisfied and there is $k_0 \in \{1, 2, 3\}$ such that $\delta_{k_0} < 1$ and for every $k = 1, 2, 3$ one can find $i_0 \in \{1, 2, 3\}$ with $|b_{i_0,k_0,k}| + |b_{k_0,i_0,k}| \neq 0$;*

Then $V^n(\mathbf{f}) \to 0$ as $n \to \infty$.

4.2. **Diagonal case.** In this subsection we are going to investigate more concrete case called diagonal operators.

We call a quadratic operator V with coefficients $\{b_{ml,k}\}$ is *diagonal* if $b_{ij,k} = 0$ for all i, j with $i \neq j$. in what follows for the sake of shortness we write b_{ik} instead of $b_{ii,k}$. Hence from (4.3) one finds

$$(V(x))_k = \sum_{i=1}^{3} b_{i,k} x_i^2. \tag{4.9}$$

Lemma 4.8. *Let V be a diagonal quadratic operator with coefficients $\{b_{ij}\}$. Assume that one holds*

$$\sum_{k=1}^{3} \max_i\{|b_{i,k}|^2\} \leq 1, \tag{4.10}$$

then (4.1) *is satisfied.*

Proof. Let us check (4.1). Take any $\mathbf{f}, \mathbf{p} \in S$, then taking into account the definition of diagonal operator and our notation we get

$$\begin{aligned} \left|\sum_{i,j=1}^{3} b_{ij,k} f_i p_j\right| &\leq \sum_{i=1}^{3} |b_{i,k}||f_i||p_i| \\ &\leq \max_i\{|b_{i,k}|\} \sum_{i=1}^{3} |f_i||p_i| \\ &\leq \max_i\{|b_{i,k}|\}\|\mathbf{f}\|\|\mathbf{p}\| \\ &\leq \max_i\{|b_{i,k}|\}, \end{aligned}$$

which implies the desired inequality. □

One can see that (4.10) is weaker than (4.2). Now denote $a_k := \max_i\{|b_{i,k}|\}$ and put

$$\gamma := \sum_{k=1}^{3} a_k^2. \tag{4.11}$$

From (4.11) we find $|V(x)_k| \leq a_k$ and hence

$$|V^2(x)_k| \leq a_k\gamma.$$

Now iterating this procedure one gets

$$|V^n(x)_k| \leq a_k\gamma^{n-1} \tag{4.12}$$

for every $n \geq 2$. From (4.10) we know that $\gamma \leq 1$. Assume that $\gamma < 1$, then (4.12) implies that $V^n(x) \to 0$ as $n \to \infty$ for every $x \in S$. Hence we can formulate the following

Theorem 4.9. *Let V be a diagonal quadratic operator* (4.9) *with coefficients $\{b_{ij}\}$. Assume that*

$$\sum_{k=1}^{3} \max_i\{|b_{i,k}|^2\} < 1, \tag{4.13}$$

then the operator has a unique stable fixed point $(0, 0, 0)$.

Now consider an example for which (4.13) is not satisfied. We shall see that in this case quadratic operator may have more than one fixed points.

4.3. **Example of diagonal quadratic operator for which** (4.13) **is not valid. Moreover, it is not KS one.** Let us consider the following diagonal quadratic operator defined by

$$\left\{\begin{array}{l}(V(x))_1 = x_1^2\\ (V(x))_2 = ax_2^2 + bx_3^2\\ (V(x))_3 = cx_3^2\end{array}\right. \tag{4.14}$$

We can immediately observe that for given operator (4.13) is not satisfied since $b_{11} = 1$.

We need one auxiliary result.

Lemma 4.10. *If* $\mathbf{f} = (f_1, f_2, f_3), \mathbf{p} = (p_1, p_2, p_3) \in S$ *then one has*

$$f_1p_1 + f_2p_2 + f_3p_3 \le 1.$$

The proof is straightforward.

Lemma 4.11. *Let*

$$\max\{a^2, b^2\} + c^2 \le 1 \tag{4.15}$$

be satisfied. Then the quadratic operator (4.14) *satisfies* (4.1).

Proof. Take any $\mathbf{f}, \mathbf{p} \in S$, then using Lemma 4.10 we have

$$\begin{aligned}\sum_{k=1}^{3}\left|\sum_{m,l=1}^{3} b_{ml,k}f_mp_l\right|^2 &= |f_1p_1|^2 + |af_2p_2 + bf_3p_3|^2 + |cf_3p_3|^2\\ &\le |f_1p_1|^2 + \max\{a^2, b^2\}(|f_2p_2| + |f_3p_3|)^2 + c^2|f_3p_3|\\ &\le (f_1p_1)^2 + \max\{a^2, b^2\}(|f_2p_2| + |f_3p_3|)^2 + c^2(|f_2p_2| + |f_3p_3|)\\ &\le (1 - |f_2p_2| - |f_3p_3|)^2 + \max\{a^2, b^2\}(|f_2p_2| + |f_3p_3|)^2 + c^2(|f_2p_2| + |f_3p_3|)\\ &\le (1 - z)^2 + \max\{a^2, b^2\}z^2 + c^2z\\ &= z^2(1 + \max\{a^2, b^2\}) + z(c^2 - 2) + 1,\end{aligned}$$

where we have put $z = |f_2p_2| + |f_3p_3|$. So, if the following inequality holds

$$z^2(1 + \max\{a^2, b^2\}) + z(c^2 - 2) + 1 \le 1 \tag{4.16}$$

then (4.2) is satisfied. The last inequality (4.16) is equivalent to

$$z(\max\{a^2, b^2\} + 1) + c^2 - 2 \le 0. \tag{4.17}$$

Taking into account $0 \le z \le 1$, we conclude that (4.17) is satisfied if one has

$$\max\{a^2, b^2\} + 1 + c^2 - 2 \le 0$$

which implies the assertion. □

The proved lemma implies that the operator (4.14) maps S into itself. Now let us examine dynamics of (4.14).

It is easy to observe that $(0, 0, 0)$, $(1, 0, 0)$ are fixed points of (4.14). Take $x \in S$ with $|x_1| = 1$, in this case $x_2 = x_3 = 0$. Then from (4.14) one finds that $V^N(x) \to (1, 0, 0)$ as $n \to \infty$.

Consider several cases with respect to the variation of parameters.

`Case (I)`. Let $|c| = 1$. Then from (4.15) one gets that $a = b = 0$. In this case, we have another fixed point $(0, 0, c)$, and $(0, 0, -c)$ is an eventually fixed point. So, if $|x_3| = 1$ then $V^n(x) \to (0, 0, c)$ as $n \to \infty$. If $x_1^2 + x_3^2 < 1$ then $V^n(x) \to (0, 0, 0)$.

`Case (II)`. Let $\max\{a^2, b^2\} = 1$. Then from (4.15) one finds $c = 0$. In this case, there is another fixed point $(0, a, 0)$ if $|a| = 1$. From (4.14) we have $(V^n(x))_2 = a(ax_2^2 + bx_3^2)^{2^{n-1}}$. So,

$ax_2^2 + bx_3^2 = 1$ then $V^n(x) \to (0, a, 0)$. If $|ax_2^2 + bx_3^2| < 1$ and $|x_1| < 1$ then $V^n(x) \to (0, 0, 0)$.

`Case (III)`. Let us assume that $\max\{a^2, b^2\} + c^2 < 1$. Then modulus of all the coefficients are strictly less the one. For the sake of simplicity denote $m = \max\{|a|, |b|\}$. From (4.14) we have $|(V(x))_2| \le m,\ |(V(x))_3| \le |c|$.

Let us denote $\kappa = m^2 + |c|^2$. Then from (4.14) one gets

$$\begin{cases} |(V(x))_2| \le m\kappa \\ |(V(x))_3| \le |c|^3 \end{cases} \tag{4.18}$$

Consequently we have

$$\begin{aligned} |(V^2(x))_2| &\le m(m^2\kappa^2 + |c|^6) \\ &\le m(m^2\kappa^2 + \kappa^2|c|^2) = m\kappa^3, \end{aligned}$$

here we have used that $|c|^2 \le \kappa$. So, by induction we may easily prove the following

$$\begin{cases} |(V^n(x))_2| \le m\kappa^{2^n-1} \\ |(V^n(x))_3| \le |c|^{2^{n+1}-1} \end{cases} \tag{4.19}$$

According to our assumption $\kappa < 1$, therefore when $|x_1| < 1$ we have $V^n(x) \to (0, 0, 0)$ $(n \to \infty)$.

Finally let us choose parameters a, b, c in such a way that corresponding linear operator Δ is not KS-operator. To do it, it is enough choose the numbers for which conditions of Corollary 3.7 are satisfied. Let us start to look to (3.17). A little calculations show that

$$\mathbf{x}_1 = (\overline{w}_1, 0, 0), \mathbf{x}_2 = (0, a\overline{w}_2, 0), \mathbf{x}_3 = (0, 0, b\overline{w}_2 + c\overline{w}_3) \tag{4.20}$$

So, from (3.11) we immediately find

$$\alpha_{2,3} = \langle \mathbf{x}_2, \mathbf{x}_3 \rangle - \langle \mathbf{x}_3, \mathbf{x}_2 \rangle = 0.$$

Hence, from the last equality with (4.20) we infer that (3.17) is reduced to

$$|a|^2|w_2|^2 + |b\overline{w}_2 + c\overline{w}_3|^2 \le |w_2|^2 + |w_3|^2 \tag{4.21}$$

Now let us estimate left hand side the expression of (4.21).

$$\begin{aligned} |a|^2|w_2|^2 + |b\overline{w}_2 + c\overline{w}_3|^2 &\le |a|^2|w_2|^2 + \Big(|b||w_2| + |c||w_3|\Big)^2 \\ &\le |a|^2|w_2|^2 + \max\{|b|^2, |c|^2\}\Big(|w_2| + |w_3|\Big)^2 \\ &\le |a|^2|w_2|^2 + 2\max\{|b|^2, |c|^2\}\Big(|w_2|^2 + |w_3|^2\Big) \end{aligned}$$

So, if we have

$$|a|^2|w_2|^2 + 2\max\{|b|^2, |c|^2\}\Big(|w_2|^2 + |w_3|^2\Big) \le |w_2|^2 + |w_3|^2 \tag{4.22}$$

then surely (4.21) is satisfied. Therefore, let us examine (4.22). From (4.22) one finds

$$\Big(1 - |a|^2 - 2\max\{|b|^2, |c|^2\}\Big)|w_2|^2 + \Big(1 - 2\max\{|b|^2, |c|^2\}\Big)|w_3|^2 \ge 0,$$

which is satisfied if one has

$$|a|^2 + 2\max\{|b|^2, |c|^2\} \leq 1. \tag{4.23}$$

Now let us look to the condition (3.18). From (4.20) direct calculations shows us that

$$\begin{cases} \mathbf{h}(\mathbf{w}) = \big(\overline{w}_2 w_3 - \overline{w}_3 w_2, 0, 0\big) \\ \gamma_{2,3} = \big(2ab|w_2|^2 + ac(\overline{w}_2 w_3 + w_2\overline{w}_3), 0, 0\big) \\ \sum\limits_{m=1}^{3} [\mathbf{x}_m, \overline{\mathbf{x}}_m] = 0 \end{cases} \tag{4.24}$$

Therefore, left hand side of (3.18) can be written as follows

$$\left\|\mathbf{h}(\mathbf{w}) - i\gamma_{2,3} + i\sum_{m=1}^{3}[\mathbf{x}_m, \overline{\mathbf{x}}_m]\right\| = \left|\overline{w}_2 w_3(1 - iac) - \overline{w}_3 w_2(1 + iac) - 2iab|w_2|^2\right|.$$

Hence, by means of (4.20) and the last equality we find that (3.18) reduces to

$$\left|\overline{w}_2 w_3(1 - iac) - \overline{w}_3 w_2(1 + iac) - 2iab|w_2|^2\right| \leq |w_2|^2 + |w_3|^2 - |a|^2|w_2|^2 - |b\overline{w}_2 + c\overline{w}_3|^2$$

Letting $w_3 = 0$ in the last inequality, one gets

$$2|ab||w_2|^2 \leq |w_2|^2(1 - |a|^2 - |b|^2)$$

which is equivalent to

$$|a| + |b| \leq 1. \tag{4.25}$$

Consequently, if we choose the numbers a and b in such a way for them (4.15) is valid, but (4.25) is not satisfied then the corresponding q.q.o. Δ would not be a Kadison-Schwartz operator. For example, let us put $c = 0$, $a = b = 1/\sqrt{3}$. Then one can see that (4.15) is satisfied, moreover (4.23) is valid as well. But (4.25) is not satisfied.

Acknowledgement

The first (F.M) named author thanks the Scientific and Technological Research Council of Turkey (TUBITAK) for support and Harran University for kind hospitality. Moreover, he and the forth named author (A.A.) also acknowledge Research Endowment Grant B (EDW B 0905-303) of IIUM and the MOSTI grant 01-01-08-SF0079.

References

[1] S. N. Bernstein, The solution of a mathematical problem related to the theory of heredity, *Uchen. Zapiski NI Kaf. Ukr. Otd. Mat.* 1924, no. 1, 83115. (Russian)
[2] L. Boltzmann, *Selected works*, Nauka, Moscow 1984. (Russian)
[3] O. Bratteli and D. W. Robertson, *Operator algebras and quantum statistical mechanics.* I, Springer, New YorkHeidelbergBerlin 1979.
[4] N. N. Ganikhodzhaev and F. M. Mukhamedov, On quantum quadratic stochastic processes, and some ergodic theorems for such processes, *Uzb. Matem. Zh.* 1997, no. 3, 820. (Russian)
[5] N.N. Ganikhodzhaev, F. M. Mukhamedov, Ergodic properties of quantum quadratic stochastic processes, *Izv. Math.* **65** (2000), 873890.
[6] U. Franz, A. Skalski, On ergodic properties of convolution operators associated with compact quantum groups, *Colloq. Math.* **113** (2008), 13–23.
[7] H. Kesten, Quadratic transformations: a model for population growth. I, II, *Adv. in Appl. Probab.* 1970, no. 2, 182, 179228.

[8] A. Kossakowski, A Class of Linear Positive Maps in Matrix Algebras, *Open Sys. & Information Dyn.* **10**(2003) 213–220.
[9] Yu. I. Lyubich, Basic concepts and theorems of the evolutionary genetics of free populations, *Russian Math. Surveys* **26**(1971).
[10] Yu. I. Lyubich, *Mathematical structures in population genetics*, Springer, Berlin 1992.
[11] W.A. Majewski, M. Marciniak, On a characterization of positive maps, *J. Phys. A: Math. Gen.* **34** (2001) 5863-5874.
[12] W.A. Majewski, M. Marciniak, On the structure of positive maps between matrix algebras, `arXiv:0705.0798`
[13] W.A. Majewski, M. Marciniak, k-decomposability of positive maps, `arXiv:quant-ph/0411035`.
[14] V. M. Maksimov, Cubic stochastic matrices and their probability interpretations, *Theory Probab. Appl.* **41** (1996), 5569.
[15] F.M. Mukhamedov, On ergodic properties of discrete quadratic dynamical system on C^*-algebras. *Method of Funct. Anal. and Topology*, **7**(2001), No.1, 63–75.
[16] F.M. Mukhamedov, On decomposition of quantum quadratic stochastic processes into layer-Markov processes defned on von Neumann algebras, *Izvestiya Math.* **68**(2004), No.5. 1009–1024.
[17] F. Mukhamedov, Dynamics of quantum quadratic stochastic operators on $M_2(C)$, In book: M. Darus, S. Owa (Eds), Proc. Inter. Symp. New Developments of Geometric Function Theory & its Appl. 10-13 Novemer 2008, Natioanl Univ. Malaysia, pp. 425–430.
[18] M.A. Nielsen, I.L. Chuang, *Quantum Computation and Quantum Information*, Cambridge University Press, Cambridge, 2000.
[19] M.B. Ruskai, S. Szarek, E. Werner, An analysis of completely positive trace-preserving maps on M_2, *Lin. Alg. Appl.* **347** (2002) 159-187.
[20] P.M. Soltan, Quantum $SO(3)$ groups and quantum group actions M_2, `arXiv:0810.0398`.
[21] S. M. Ulam, *A collection of mathematical problems*, Interscience, New YorkLondon 1960.
[22] S. S. Vallander, On the limit behaviour of iteration sequences of certain quadratic transformations *Soviet Math. Dokl.* **13** (1972), 123126.
[23] S.L. Woronowicz, Compact matrix pseudogroups. *Comm. Math. Phys.* **111** (1987), 613-665.

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